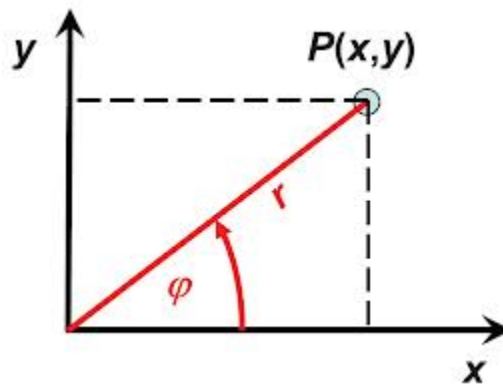


**Phys 410**  
**Fall 2013**  
**Lecture #2 Summary**  
**5 September, 2013**

We considered Newton's second law of motion in two-dimensional polar coordinates ( $r, \varphi$ ):  $m\ddot{\vec{r}} = F_{net}$ . The basic issue is that the polar unit vectors  $\hat{r}$  and  $\hat{\varphi}$  change direction as the particle moves about the plane. Note that  $\hat{r}$  is defined as the direction of increasing  $r$  coordinate at fixed  $\varphi$  coordinate, and similarly for  $\hat{\varphi}$ . Both unit vectors are functions of the angular coordinate  $\varphi$ . They can be written in terms of the directionally-invariant Cartesian unit vectors  $\hat{i}$  and  $\hat{j}$  as  $\hat{r} = \cos\varphi \hat{i} + \sin\varphi \hat{j}$ , and  $\hat{\varphi} = -\sin\varphi \hat{i} + \cos\varphi \hat{j}$ . A moving particle is parameterized by the time-dependent functions  $r(t)$  and  $\varphi(t)$ , yielding a time-varying pair of unit vectors  $d\hat{r}/dt = \dot{\varphi} \hat{\varphi}$  and  $d\hat{\varphi}/dt = -\dot{\varphi} \hat{r}$ . Starting from the coordinate vector  $\vec{r} = r\hat{r}$ , one can take two derivatives to find the rather complicated result  $\vec{a} = (\ddot{r} - r\dot{\varphi}^2)\hat{r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\varphi}$ .



We considered projectile motion with air resistance. The drag force is directed opposite to the instantaneous velocity and depends on the speed as  $\vec{f} = -f(v)\hat{v}$ , where  $f(v) = 0 + bv + cv^2$ , and  $\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{v}$  is the direction unit vector of the velocity. The linear term  $bv$  in  $f(v)$  arises from viscous friction as the object moves through the fluid (air, water, honey, etc.), and depends on the linear dimension of the object,  $D$ . The quadratic term  $cv^2$  arises from accelerating the fluid out of the way of the object, and depends on the cross-sectional area of the object,  $D^2$ . The ratio of the two forces is  $\frac{f_{quad}}{f_{lin}} = 1.56 \times 10^3 \frac{s}{m^2} Dv$  for a spherical object of diameter  $D$  moving through air at STP (20°C, 1 atmosphere pressure). This ratio is directly related to the Reynold's number in hydrodynamics. We found that the quadratic force dominates the linear force for a free-falling skydiver, and a golf ball flying through the air. The two forces are comparable for a falling rain-drop, and the linear force dominates for a small oil particle falling in air in the Millikan oil drop [experiment](#).

An object falling under the influence of gravity and the linear drag force obeys the equation of motion:  $m\vec{\dot{v}} = m\vec{g} - b\vec{v}$ . In a Cartesian coordinate system where  $x$  is horizontal and  $y$  points downward, the vector equation breaks cleanly in to two un-coupled scalar first-order equations:  $m\dot{v}_x = 0 - bv_x$ , and  $m\dot{v}_y = mg - bv_y$ . The horizontal and vertical equations can be solved separately, and the full solution can be constructed later.

The horizontal equation can be written as  $\dot{v}_x = -\frac{b}{m}v_x = -\frac{1}{\tau}v_x$ , where we have defined a characteristic time of  $\tau \equiv m/b$ . Integrating the equation after separating variables, one finds  $v_x(t) = v_{x0}e^{-t/\tau}$ , where  $v_x(0) = v_{x0}$  is the initial velocity. The velocity simply decays to zero after its initial kick, and the “1/e decay time” is  $\tau$ . The velocity equation can be integrated to find the position of the particle moving horizontally under the influence of linear drag:  $x(t) - x(0) = x_\infty(1 - e^{-t/\tau})$ , where  $x_\infty = v_{x0}\tau$ . The particle only moves a finite distance  $x_\infty$  before effectively coming to rest.

We then considered vertical motion of a point-like object subjected to a linear drag force. Newton’s second law of motion for the vertical component of motion is given by  $m\dot{v}_y = mg - bv_y$ , with  $y$ -positive in the downward direction. Starting from rest, the particle will initially experience no drag force. As it accelerates downward, the drag force will increase steadily. At some point the drag force will equal the force of gravity, and the acceleration will cease. The object will be moving at the terminal velocity from that time onward, given by the balance of the two forces as  $v_{ter} = \frac{gm}{b} = \tau g$ . The equation of motion can be written as  $m\dot{v}_y = b(v_y - v_{ter})$ , with a solution  $v_y(t) = v_{y0}e^{-t/\tau} + v_{ter}(1 - e^{-t/\tau})$ . This equation shows that the velocity starts as  $v_{y0}$  and then switches over to  $v_{ter}$  as time evolves. The time scale for this crossover is  $\tau = m/b$ , the characteristic time governing the horizontal motion as well. The velocity equation can be integrated to find the vertical position as a function of time:  $y(t) - y(0) = v_{ter}t - (v_{y0} - v_{ter})\tau(1 - e^{-t/\tau})$ .

Combining the solutions for the  $x$ -motion and  $y$ -motion, we can now construct a solution (trajectory) for general motion in the  $xy$ -plane for a particle experiencing linear drag. Solving the  $x(t)$  equation from above for  $t$  in terms of  $x$ , and putting this into the  $y(t)$  equation above, we can find the trajectory of the particle:  $y(x) = \frac{v_{y0} + v_{ter}}{v_{x0}}x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)$  (now it is assumed that positive  $y$  is in the “up” direction). The trajectory in the absence of drag is given by  $y(x) = \frac{v_{y0}}{v_{x0}}x - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2$ . The main difference is in the second term. The logarithm is negative for all non-zero values of  $x$ , and has a negative divergence as the argument goes to zero. This occurs when  $x = \tau v_{x0}$ , showing that the horizontal range of the projectile is limited by this value. Hence when Felix stepped out of the capsule and gave himself an initial horizontal velocity ( $v_{x0}$ ), his net displacement in the horizontal direction was limited (assuming that a linear drag force acted upon him at least in the initial part of the fall).